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On the stability of viscous flow between eccentric rotating cylinders

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The stability of viscous flow between eccentric cylinders is analysed for the case in which the inner cylinder rotates while the outer cylinder remains stationary, and where the difference in radii of the cylinders is small in comparison with their mean radius. The linearised equations governing the marginal stability of axially periodic disturbances are derived in general for the case where the cylinders are infinitely long, and are solved approximately to give estimates of the critical Taylor number at which vortex flow occurs for a range of relative eccentricity of the cylinders.

The results give an upper bound to the stability boundary, and certain results of DiPrima are used to establish a lower bound, and consequently the stability boundary is well established for eccentricity ratios less than about 0·6. One important conclusion is that for a considerable range of eccentricity ratio the flow is less stable than when the cylinders are concentric.

1. Introduction

The stability of viscous flow between rotating cylinders was first analysed in detail by Taylor (1923) for the case where the cylinders are concentric and where the gap between them is small compared to their mean radius. Since then numerous authors, including Pellew & Southwell (1940), Chandrasekhar (1954) and DiPrima (1955) have published more refined solutions of the eigenvalue problem. For large relative gap widths the problem has received attention from Chandrasekhar (1958), Chandrasekhar & Elbert (1962) and from Sparrow, Munro & Jonsson (1964). The effects of the superposition of a circumferential pressure gradient in the annulus have been examined by DiPrima (1959).

For eccentric cylinders, however, very few references are to be found in the literature. Cole (1957) and Kamal (1966) have published a few experimental results, and DiPrima (1963) has considered the problem theoretically, assuming local stability by comparison with concentric theory with circumferential pressure gradients. The main difficulty arising in the eccentric problem is of course that the fluid motion must be considered in three dimensions, in contrast with the concentric case which is axisymmetric.

The intention of this paper is to derive the marginal stability problem in a co-ordinate system suitable for eccentric rotating cylinders, and to present approximate solutions of the resulting eigenvalue problem in the restricted, but important case of a very small clearance ratio.

2. Co-ordinate system and equations of motion

For steady flow of an incompressible, isoviscous fluid, the Navier–Stokes and continuity equations can be expressed in the following vector form:

$$\tfrac{1}{2}\nabla(\mathbf{q}.\mathbf{q})+(\nabla\times\mathbf{q})\times\mathbf{q} = -(1/\rho)\nabla p+\nu\{\nabla(\nabla.\mathbf{q})-\nabla\times\nabla\times\mathbf{q}\}, \tag{1}$$

$$\nabla.\mathbf{q} = 0, \tag{2}$$

where $\mathbf{q}$, p, ρ and ν are respectively the velocity vector, pressure, density and kinematic viscosity of the fluid.

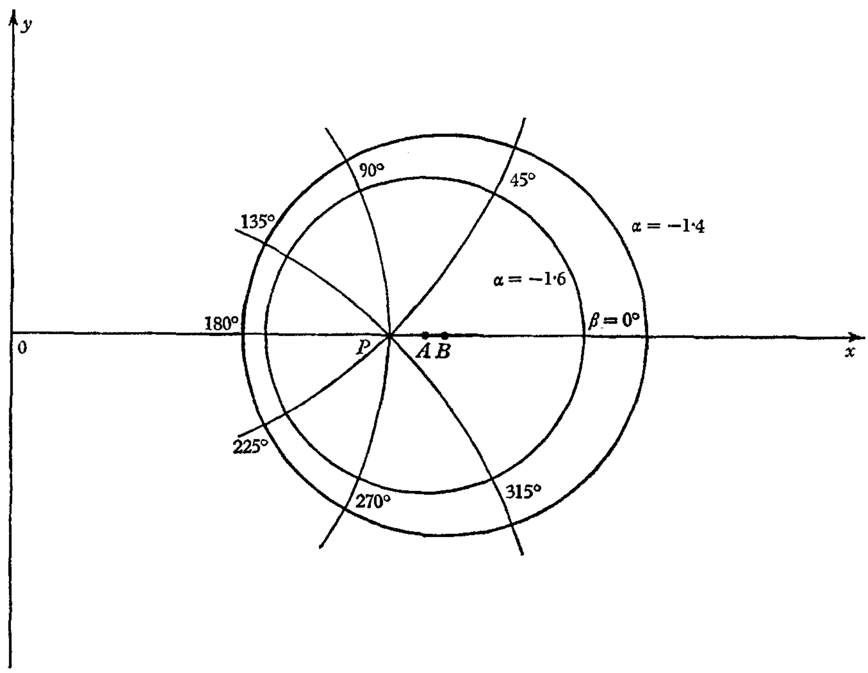

FIGURE 1. The bipolar co-ordinate system. Clearance ratio, ψ = 0·25; eccentricity ratio, ϵ = 0·42.

The general orthogonal co-ordinate system for the geometry of eccentric paraxial cylinders is a cylindrical bipolar system. In such a system a pair of parallel eccentric cylinders, with arbitrary clearance and eccentricity ratios, is represented by two discrete values of one of the co-ordinates (see figure 1). The system (α, β, z) used in the following analysis has the additional advantage that the Lamé coefficients h_α and h_β are equal at each point.

The transformation from Cartesian co-ordinates (x, y, z) to the cylindrical bi-polar system (α, β, z) is given by

$$x = \frac{-a\sinh\alpha}{\cosh\alpha-\cos\beta}, \quad y = \frac{a\sin\beta}{\cosh\alpha-\cos\beta}, \quad z = z. \tag{3}$$

The Lamé coefficients are found to be $(h, h, 1)$ where

$$h = \frac{a}{\cosh\alpha - \cos\beta}, \tag{4}$$

where, referring to figure 1, a is the length from the Cartesian origin O to the pole P of the bipolar system.

If the inner cylinder, radius R_1, and the outer cylinder, radius R_2, correspond to $\alpha = \alpha_1$ and $\alpha = \alpha_2$ respectively, and the clearance and eccentricity ratios are denoted by ψ and ϵ respectively, it may be shown that

$$\left.\begin{aligned} R_1 &= \frac{-a}{\sinh\alpha_1}, \quad R_2 = \frac{-a}{\sinh\alpha_2};\\ \psi &= \frac{R_2 - R_1}{R_2} = \frac{C}{R_2} = \frac{\sinh\alpha_1 - \sinh\alpha_2}{\sinh\alpha_1};\\ \epsilon &= \frac{OB - OA}{C} = \frac{\sinh(\alpha_1 - \alpha_2)}{\sinh\alpha_1 - \sinh\alpha_2}. \end{aligned}\right\} \tag{5}$$

In this co-ordinate system, if (U, V, W) are the velocity components and P denotes the pressure, the equations of motion (1) and (2) are

$$\begin{aligned} &\frac{U}{h}\frac{\partial U}{\partial\alpha} + \frac{V}{h}\frac{\partial U}{\partial\beta} + W\frac{\partial U}{\partial z} + \frac{V^2\sinh\alpha}{a} - \frac{UV\sin\beta}{a} = -\frac{1}{\rho h}\frac{\partial P}{\partial\alpha}\\ &\quad + \nu\left\{\frac{1}{h^2}\frac{\partial^2 U}{\partial\alpha^2} + \frac{1}{h^2}\frac{\partial^2 U}{\partial\beta^2} + \frac{\partial^2 U}{\partial z^2} - \frac{(\cosh\alpha + \cos\beta)U}{ah} + \frac{2\sinh\alpha}{ah}\frac{\partial V}{\partial\beta} - \frac{2\sin\beta}{ah}\frac{\partial V}{\partial\alpha}\right\}, \end{aligned} \tag{6a}$$

$$\begin{aligned} &\frac{U}{h}\frac{\partial V}{\partial\alpha} + \frac{V}{h}\frac{\partial V}{\partial\beta} + W\frac{\partial V}{\partial z} - \frac{UV\sinh\alpha}{a} + \frac{U^2\sin\beta}{a} = -\frac{1}{\rho h}\frac{\partial P}{\partial\beta}\\ &\quad + \nu\left\{\frac{1}{h^2}\frac{\partial^2 V}{\partial\alpha^2} + \frac{1}{h^2}\frac{\partial^2 V}{\partial\beta^2} + \frac{\partial^2 V}{\partial^2 z} - \frac{(\cosh\alpha + \cos\beta)V}{ah} - \frac{2\sinh\alpha}{ah}\frac{\partial U}{\partial\beta} + \frac{2\sin\beta}{ah}\frac{\partial U}{\partial\alpha}\right\}, \end{aligned} \tag{6b}$$

$$\frac{U}{h}\frac{\partial W}{\partial\alpha} + \frac{V}{h}\frac{\partial W}{\partial\beta} + W\frac{\partial W}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial z} + \nu\left\{\frac{1}{h^2}\frac{\partial^2 W}{\partial\alpha^2} + \frac{1}{h^2}\frac{\partial^2 W}{\partial\beta^2} + \frac{\partial^2 W}{\partial z^2}\right\}, \tag{6c}$$

$$\frac{1}{h^2}\frac{\partial}{\partial\alpha}(hU) + \frac{1}{h^2}\frac{\partial}{\partial\beta}(hV) + \frac{\partial W}{\partial z} = 0. \tag{7}$$

3. The linear stability problem

In order to carry out a linearized stability analysis of flow between eccentric rotating cylinders, an explicit laminar solution for the flow must be available. This is a matter of some difficulty, and it is clear that a two-dimensional laminar flow, implying no axial variations, must be considered. At this stage it is assumed that such a solution (U', V', O', P') is available and the linear stability problem is formulated, the problem of determining the laminar flow being considered later in § 5.

The equations of marginal stability are obtained from (6), (7) by perturbing the laminar flow (U', V', O, P') with an infinitesimal axially periodic disturbance, subtracting out the equations satisfied by the laminar flow, and retaining only linear terms in the disturbance velocity components. The perturbed motion is therefore taken as

$$\left.\begin{aligned} U &= U'+u'\cos\lambda z, \quad V = V'+v'\cos\lambda z; \\ W &= w'\sin\lambda z, \quad P = P'+p'\cos\lambda z; \end{aligned}\right\} \tag{8}$$

where u', v', w' and p' are independent of z (as are U', V' and P'), and λ is some unknown axial wave-number.

The resulting equations, defining the exact linear stability problem, are

$$\begin{aligned} &\frac{1}{h}\frac{\partial}{\partial\alpha}(U'u')+\frac{V'}{h}\frac{\partial u'}{\partial\beta}+\frac{v'}{h}\frac{\partial U'}{\partial\beta}+\frac{2V'v'\sinh\alpha}{a}-\frac{(U'v'+V'u')\sin\beta}{a} \\ &= -\frac{1}{\rho h}\frac{\partial p'}{\partial\alpha}+\nu\left\{\frac{1}{h^2}\frac{\partial^2 u'}{\partial\alpha^2}+\frac{1}{h^2}\frac{\partial^2 u'}{\partial\beta^2}-\left(\lambda^2+\frac{\cosh\alpha+\cos\beta}{ah}\right)u'+\frac{2\sinh\alpha}{ah}\frac{\partial v'}{\partial\beta}-\frac{2\sin\beta}{ah}\frac{\partial v'}{\partial\alpha}\right\}, \end{aligned} \tag{9a}$$

$$\begin{aligned} &\frac{U'}{h}\frac{\partial v'}{\partial\alpha}+\frac{u'}{h}\frac{\partial V'}{\partial\alpha}+\frac{1}{h}\frac{\partial}{\partial\beta}(V'v')-\frac{(U'v'+V'u')\sinh\alpha}{a}+\frac{2U'u'\sin\beta}{a} \\ &= -\frac{1}{\rho h}\frac{\partial p'}{\partial\beta}+\nu\left\{\frac{1}{h^2}\frac{\partial^2 v'}{\partial\alpha^2}+\frac{1}{h^2}\frac{\partial^2 v'}{\partial\beta^2}-\left(\lambda^2+\frac{\cosh\alpha+\cos\beta}{ah}\right)v'-\frac{2\sinh\alpha}{ah}\frac{\partial u'}{\partial\beta}+\frac{2\sin\beta}{ah}\frac{\partial u'}{\partial\alpha}\right\}, \end{aligned} \tag{9b}$$

$$\frac{U'}{h}\frac{\partial w'}{\partial\alpha}+\frac{V'}{h}\frac{\partial w'}{\partial\beta} = \frac{\lambda p'}{\rho}+\nu\left\{\frac{1}{h^2}\frac{\partial^2 w'}{\partial\alpha^2}+\frac{1}{h^2}\frac{\partial^2 w}{\partial\beta^2}-\lambda^2 w\right\}, \tag{9c}$$

$$\frac{1}{h^2}\frac{\partial}{\partial\alpha}(hu')+\frac{1}{h^2}\frac{\partial}{\partial\beta}(hv')+\lambda w' = 0. \tag{10}$$

The boundary conditions are

$$\left.\begin{aligned} u' = v' = w' = 0 \quad \text{for} \quad \alpha = \alpha_1 \quad \text{and} \quad \alpha = \alpha_2, \\ u', v', w' \text{ and } p' \text{ are periodic in } \beta. \end{aligned}\right\} \tag{11}$$

As with the concentric (Taylor vortex) case considerable simplification of the problem will result if the clearance ratio is small compared to unity. Furthermore, it will be seen that only if this condition is enforced does a relatively simple form for the laminar flow become available.

In the following section the framework of the small gap analysis is set up and the linear stability problem is non-dimensionalized into the required small-gap form.

4. Small gap analysis

If $\psi \ll 1$ it seems reasonable to suppose that a term of $O(\psi)$ may safely be neglected in comparison with a similar term of $O(1)$, and using this criterion the following may be deduced

$$\left.\begin{aligned} \sinh\alpha_1 \simeq \sinh\alpha_2 &\simeq -\frac{(1-\epsilon^2)^{\frac{1}{2}}}{\epsilon}, \\ \cosh\alpha_1 \simeq \cosh\alpha_2 &\simeq \frac{1}{\epsilon}, \\ \sinh(\alpha_1-\alpha_2) \simeq (\alpha_1-\alpha_2) &\simeq -\psi(1-\epsilon^2)^{\frac{1}{2}}, \end{aligned}\right\} \tag{12}$$

$$a \simeq \frac{R_2(1-\epsilon^2)^{\frac{1}{2}}}{\epsilon}; \quad h \simeq \frac{R_2(1-\epsilon^2)^{\frac{1}{2}}}{1-\epsilon\cos\beta} \tag{13}$$

and hence, in this approximation, h is independent of α.

A more convenient small parameter t than ψ is defined by

$$t = -(\alpha_1 - \alpha_2) = \psi(1-\epsilon^2)^{\frac{1}{2}} \tag{14}$$

so that t is a positive number of $O(\psi)$.

A scaled co-ordinate ζ measuring distance across the gap between the cylinders may now be defined by:

$$\alpha = \alpha_1 + t\zeta \tag{15}$$

so that the inner and outer cylinders are now parametrized by $\zeta = 0$ and $\zeta = 1$ respectively.

The object of using this scaled co-ordinate ζ is that orders of magnitude of terms may be readily computed as powers of the small parameter t, and that terms can be retained or rejected according to the criterion that a term of $O(t)$ is negligible compared to a similar term of $O(1)$.

It is necessary at this stage to derive one important result concerning the relative magnitudes of the laminar velocity components (U', V'). In the co-ordinate system (ζ, β) the equation of continuity (7) for the two-dimensional flow is

$$\frac{h}{t}\frac{\partial U'}{\partial \zeta} + \frac{\partial}{\partial \beta}(hV') = 0 \tag{16}$$

and it can be deduced that, if V' is of $O(1)$ then U' is of $O(t)$.

In deriving the linear stability problem in small gap form it is convenient to use non-dimensional variables defined as follows, where V_0 is the velocity of the inner cylinder:

$$u' = V_0 u, \quad v' = V_0 v, \quad w' = V_0 w, \quad U' = tV_0 U, \quad V' = V_0 V, \tag{17}$$

$$\lambda = k/c, \quad h = R_2 H, \quad R = V_0 C/\nu, \quad p' = \rho V_0^2 p, \quad P' = \rho V_0^2 P. \tag{18}$$

Equations (9), (10), expressed in small gap form, now become

$$\left\{\frac{\partial^2}{\partial\zeta^2} - k^2H^2(1-\epsilon^2)\right\}u = RH(1-\epsilon^2)^{\frac{1}{2}}\frac{\partial p}{\partial \zeta} + R\psi H^2(1-\epsilon^2)\left\{\frac{1}{H}\frac{\partial}{\partial\zeta}(Uu) + \frac{V}{H^2}\frac{\partial}{\partial\beta}(Hu) - 2Vv\right\}, \tag{19a}$$

$$\left\{\frac{\partial^2}{\partial\zeta^2} - k^2H^2(1-\epsilon^2)\right\}v = R\psi H(1-\epsilon^2)\frac{\partial p}{\partial\beta} + RH(1-\epsilon^2)^{\frac{1}{2}}\frac{\partial V}{\partial\zeta}u, \tag{19b}$$

$$\left\{\frac{\partial^2}{\partial\zeta^2} - k^2H^2(1-\epsilon^2)\right\}w = -RH^2k(1-\epsilon^2)p + R\psi H(1-\epsilon^2)\left\{U\frac{\partial w}{\partial\zeta} + V\frac{\partial w}{\partial\beta}\right\}, \tag{19c}$$

$$\frac{1}{H(1-\epsilon^2)^{\frac{1}{2}}}\frac{\partial u}{\partial\zeta} + kw = 0. \tag{20}$$

p and w can be eliminated from this set of equations to give

$$\left\{\frac{\partial^2}{\partial\zeta^2} - k^2H^2(1-\epsilon^2)\right\}v = RH(1-\epsilon^2)^{\frac{1}{2}}\frac{\partial V}{\partial\zeta}u, \tag{21a}$$

$$\left\{\frac{\partial^2}{\partial\zeta^2}-k^2H^2(1-\epsilon^2)\right\}^2 u = 2k^2R\psi(1-\epsilon^2)^2H^4$$
$$\times\left\{Vv-\frac{1}{2H}\frac{\partial}{\partial\zeta}(Uu)-\frac{V}{2H^2}\frac{\partial}{\partial\beta}(Hu)+\frac{1}{2K^2H^2(1-\epsilon^2)}\frac{\partial}{\partial\zeta}\left[U\frac{\partial}{\partial\zeta}+V\frac{\partial}{\partial\beta}\right]\left[\frac{1}{H}\frac{\partial u}{\partial\zeta}\right]\right\}. \tag{21b}$$

Equations (21*a*) and (21*b*) define the linear stability problem for flow between eccentric rotating cylinders in the small gap case. The boundary conditions on u and v are deduced from (11) and (20)

$$u = \frac{\partial u}{\partial\zeta} = v = 0 \quad \text{for} \quad \zeta = 0 \quad \text{and} \quad \zeta = 1, \tag{22}$$

u and v are periodic in β.

5. The laminar solution

The problem of laminar flow between eccentric cylinders for the case where the inner rotates and the outer is stationary has been considered by Kamal (1966), using the same bipolar system as is used here. The method adopted was to obtain, first, a solution where the non-linear inertia terms are neglected, which in the general case reduces to solving the biharmonic equation for a stream function, and subsequently to estimate an inertia correction by iterating from this stream function using the full non-linear equations.

A similar procedure is used here using the small gap form of the equations of motion, which eliminates the necessity of using the stream function. Using the non-dimensional form (17), (18), the equations of motion reduce to

$$\frac{\partial P}{\partial\zeta} = 0, \tag{23a}$$

$$\frac{\partial^2 V}{\partial\zeta^2}-R\psi(1-\epsilon^2)H\frac{\partial P}{\partial\beta} = R\psi(1-\epsilon^2)H\left\{U\frac{\partial V}{\partial\zeta}+V\frac{\partial V}{\partial\beta}\right\}, \tag{23b}$$

$$H\frac{\partial U}{\partial\zeta}+\frac{\partial}{\partial\beta}(HV) = 0. \tag{23c}$$

If the outer cylinder is stationary the boundary conditions are

$$\left.\begin{array}{c} U = 0 \quad \text{for} \quad \zeta = 0, 1, \quad V = 1 \quad \text{for} \quad \zeta = 0, \quad V = 0 \quad \text{for} \quad \zeta = 1, \\ U, V, P \text{ are periodic in } \beta. \end{array}\right\} \tag{24}$$

Since (23*b*) is non-linear in U and V a solution of (23) subject to (24) is unobtainable in closed analytic form. However, if the effect of these non-linear terms is sufficiently small a good approximation to the exact solution will be obtained with the following procedure. The solution is expressed as the sum of two parts, i.e. $U = U_1+U_2$, $V = V_1+V_2$, $P = P_1+P_2$ where U_1, V_1 and P_1 satisfy (23), wherein the non-linear terms are neglected, together with the boundary conditions (24), and where U_2, V_2 and P_2 satisfy (23) with the non-linear terms calculated using the known functions U_1, V_1 (instead of U, V), together with zero velocity boundary conditions. On adoption of this procedure the following relevant variables can be derived:

$$U_1 = -\frac{1}{H}\frac{\partial H}{\partial \beta}\zeta(1-\zeta)^2, \tag{25a}$$

$$V_1 = (1-\zeta)\left\{1-3\zeta\left(1-\frac{H^*}{H}\right)\right\}, \tag{25b}$$

where

$$H^* = \frac{(1-\epsilon^2)^{\frac{1}{2}}}{1+\frac{1}{2}\epsilon^2}, \tag{26}$$

$$V_2 = -R\psi(1-\epsilon^2)\frac{\partial H}{\partial \beta}\frac{(\zeta-\zeta^2)}{420}(2-3H_0)\{6(1-H_0) + 21\zeta(1+H_0) - 7\zeta^2(17-3H_0) + 42\zeta^3(3-2H_0) - 42\zeta^4(1-H_0)\}, \tag{27}$$

where

$$H_0 = H^*/H.$$

In order to estimate the magnitude of the inertia correction velocity component V_2 it is necessary to assign a numerical value to $R\psi$. It is known (e.g. Chandrasekhar 1954) that at marginal stability for the concentric small gap case (outer cylinder stationary), the value of $R^2\psi$ is 1695, whereas the exact calculation by Sparrow, Munro & Jonsson where the radius ratio $R_1/R_2 = 0{\cdot}95$ (i.e. $\psi = 1/20$) indicates that $R^2\psi = 1801$. Hence, for $\psi = 1/20$, the small gap approximation leads to an error of some 6 % in the determination of the critical Taylor number. For this clearance ratio a representative value for $R\psi$ near the expected critical Reynolds number will be $R\psi \simeq 10$, and using this value it may be calculated from equation (27) that the maximum values of V_2 range from $\sim 1/3$ % of the velocity of the inner cylinder for $\epsilon = 0{\cdot}1$ up to ~ 5 % for $\epsilon = 0{\cdot}9$. Hence it may confidently be expected that the iteration method adopted to determine V_2 from equations (23) yields an accurate inertia correction. More important, however, it seems a reasonable conclusion that the omission of the inertia correction invokes errors no more serious than those resulting from the normal small gap analysis. It is therefore, assumed that in the following analysis, the laminar flow (U, V) is given to sufficient accuracy by (U_1, V_1) as defined in (25), (26).

6. The polar form of the linear stability problem

Returning to the linear stability problem as defined by (21), (22) it is convenient to transform the equations to a new angular co-ordinate ϕ, the polar angle based on the centre of the stationary cylinder, $\phi = 0$ corresponding to $\beta = 0$

$$\phi = \int_0^\beta H\,d\beta, \tag{28a}$$

from which may be deduced

$$\frac{\partial}{\partial \beta} = H\frac{\partial}{\partial \phi}, \quad H = \frac{1+\epsilon\cos\phi}{(1-\epsilon^2)^{\frac{1}{2}}}. \tag{28b}$$

The linear stability problem in polar form is therefore

$$\left\{\frac{\partial^2}{\partial \zeta^2} - k^2H^2(1-\epsilon^2)\right\}v = RH(1-\epsilon^2)^{\frac{1}{2}}\frac{\partial V}{\partial \zeta}u, \tag{29a}$$

$$\left\{\frac{\partial^2}{\partial\zeta^2}-k^2H^2(1-\epsilon^2)\right\}^2 u = 2k^2R\psi(1-\epsilon^2)^2H^4\left\{Vv-\frac{1}{2H}\frac{\partial}{\partial\zeta}(Uu)-\frac{V}{2H}\frac{\partial}{\partial\phi}(Hu)\right.$$
$$\left.+\frac{1}{2k^2H^2(1-\epsilon^2)}\frac{\partial}{\partial\zeta}\left[U\frac{\partial}{\partial\zeta}+HV\frac{\partial}{\partial\phi}\right]\left[\frac{1}{H}\frac{\partial u}{\partial\zeta}\right]\right\}. \tag{29b}$$

The boundary conditions are

$$u=\frac{\partial u}{\partial\zeta}=v=0 \quad\text{for}\quad \zeta=0 \quad\text{and}\quad \zeta=1, \tag{30}$$

u, v are periodic in ϕ. U and V are given by:

$$U = -\frac{\partial H}{\partial\phi}\zeta(1-\zeta)^2, \tag{31a}$$

$$V = (1-\zeta)\left\{1-3\zeta\left(1-\frac{H^*}{H}\right)\right\}, \tag{31b}$$

where
$$H^* = \frac{(1-\epsilon^2)^{\frac{1}{2}}}{1+\frac{1}{2}\epsilon^2}.$$

7. Approximate solution of the equations

The pair of equations (29) subject to the boundary conditions (30) constitutes an extremely complicated eigenvalue problem with two dependent variables u and v, and two independent variables ζ and ϕ. For given ϵ, k and ψ the equations admit solutions only for certain discrete values of the eigenvalue R, and the minimum value R_c as k varies gives the stability boundary at each value of ϵ and ψ.

Without further simplification it appears uneconomic to attempt a general solution of the problem. However, since the problem is already confined to the small gap, an appeal to certain results from concentric theory indicates that considerable simplification might still be made without too great a loss in accuracy.

For the concentric small gap case, Chandrasekhar (1954) quotes results from which the relative magnitudes of u and v may be determined. For $\zeta=\frac{1}{2}$, i.e. midway between the cylinders, it can be calculated that

$$\left|\frac{u(\frac{1}{2})}{v(\frac{1}{2})}\right| \simeq \frac{20}{R_c}$$

and for clearance ratios less than $\psi=\frac{1}{50}$ it can be deduced that

$$\left|\frac{u(\frac{1}{2})}{v(\frac{1}{2})}\right| < \frac{1}{15}.$$

It seems reasonable to assume that this will also be true in the eccentric case, at least for small eccentricity ratios, and hence if u is redefined as Ru and a Taylor number T is defined by $T=2R^2\psi$ equations (29) may be written

$$\left\{\frac{\partial^2}{\partial\zeta^2}-k^2H^2(1-\epsilon^2)\right\}v = H(1-\epsilon^2)^{\frac{1}{2}}\frac{\partial V}{\partial\zeta}u, \tag{32a}$$

$$\left\{\frac{\partial^2}{\partial\zeta^2}-k^2H^2(1-\epsilon^2)\right\}^2 u = k^2T(1-\epsilon^2)^2H^4Vv(1+F), \tag{32b}$$

where F is a complicated differential function of u, v, U, V, R, ζ and ϕ but whose numerical magnitude is expected to be small. The approximate solution obtained in this paper proceeds on the assumption that F is negligible in comparison to unity. A complete justification of this assumption and an estimate of the error involved will obviously require the solution of the complete form of the equations. It might be noted, however, that the neglected terms are somewhat similar in nature (i.e. convective accelerations) to the non-linear terms in the equations for the laminar flow (§5) where despite the considerable magnitude of the coefficients of the terms, their effect on the viscous solutions was negligible.

When the function F is omitted from (32*b*) it will be observed that there are now no circumferential differential terms in the equations and it would be possible to solve the equations regarding ϕ as a variable parameter so that the resultant eigenvalues would be functions of angular position. This was the basis of the calculations by DiPrima (1963). However, it may be observed experimentally that local instability does not occur and that vortex flow appears instantaneously in the whole circumferential gap between the cylinders. For this reason, and also because it would certainly be necessary if F was retained, ϕ is regarded as an independent variable in the eigenfunctions of the problem.

The method of approximate solution used in this work is an adaption of Galerkin's method which was used by DiPrima (1961) on a somewhat similar pair of equations resulting from a consideration of the stability of flow between concentric cylinders when disturbances are not restricted to be two-dimensional.

For the present problem it is supposed that u and v are expanded into series sums of functions having the following properties: (i) each set of functions, i.e. one each for u and v is complete in the region of the (ζ, ϕ)-plane in which the solution is to be valid; (ii) each of the functions satisfies the boundary conditions of the velocity components of which it is part; (iii) each function is multiplied by an arbitrary amplitude coefficient.

If all these conditions hold then it is reasonable to expect that it is possible to represent the solution in the form prescribed, with the values of the coefficients determined in some manner. The Galerkin method provides the process whereby these coefficients are found. The procedure involves making the error in each equation orthogonal in the (ζ, ϕ) region to each component function of the velocity being determined. In this problem the first equation is regarded as a differential equation to determine v with u as a known function and the second as an equation to determine u with v as the known function. This results in an infinite set of homogenous linear equations in the coefficients, the condition for the existence of a non-trivial solution being the vanishing of an infinite order determinant involving the eigenvalue T, and the parameters ϵ and k. In practice the series representations for u and v are limited to the first few terms, thereby reducing the order of the determinantal equation. It seems clear that this procedure can be used to give successive upper bounds to the eigenvalue, since the number of arbitrary constants in the series influences the ability of the series to conform to the true eigenfunctions, the more constants being free, the better the conformity. This conclusion is borne out by the results obtained.

If n-term series are used to represent u and v, the determinantal equation is a polynomial in T of order n whose coefficients are functions of ϵ and k. The results of this paper were obtained using a KDF 9 computer, using an Algol program to do the following: (i) choose a value of ϵ; (ii) choose a suitable range of values of k; (iii) for each k determine the numerical values of the coefficients of the polynomial in T; (iv) determine the minimum positive root T of the polynomial equations; (v) estimate T_0 and k_0, the minimum value of T and the corresponding value of k.

It will be noted that each term in the operations of each equation is symmetric about $\phi = 0$, which implies that antisymmetric and symmetric solutions are completely independent, and that the corresponding eigenvalues can be separately determined. The results of this paper were obtained by using five symmetric and two antisymmetric representations for u and v. They are:

$$(a)\quad u = A_1(\zeta-\zeta^2)^2,\qquad v = B_1(\zeta-\zeta^2);$$

$$(b)\quad u = [A_1+A_2(1-2\zeta)](\zeta-\zeta^2)^2,\qquad v = [B_1+B_2(1-2\zeta)](\zeta-\zeta^2);$$

$$(c)\quad u = [A_1+A_3(1+\epsilon\cos\phi)](\zeta-\zeta^2)^2,\qquad v = [B_1+B_3(1+\epsilon\cos\phi)](\zeta-\zeta^2);$$

$$(d)\quad u = [A_1+A_2(1-2\zeta)+A_3(1+\epsilon\cos\phi)](\zeta-\zeta^2)^2,\qquad v = [B_1+B_2(1-2\zeta)+B_3(1+\epsilon\cos\phi)](\zeta-\zeta^2);$$

$$(e)\quad u = [A_1+A_2(1-2\zeta)+[A_3+A_4(1-2\zeta)](1+\epsilon\cos\phi)](\zeta-\zeta^2)^2,\qquad v = [B_1+B_2(1-2\zeta)+[B_3+B_4(1-2\zeta)](1+\epsilon\cos\phi)](\zeta-\zeta^2);$$

$$(f)\quad u = C_1\sin\phi(\zeta-\zeta^2)^2,\qquad v = D_1\sin\phi(\zeta-\zeta^2);$$

$$(g)\quad u = [C_1+C_2(1-2\zeta)]\sin\phi\,(\zeta-\zeta^2)^2,\qquad v = [D_1+D_2(1-2\zeta)]\sin\phi(\zeta-\zeta^2).$$

These series are all of the form:

$$u = (\zeta-\zeta^2)^2\sum_n A_n\Phi_n(\zeta,\phi),$$

$$v = (\zeta-\zeta^2)\sum_n B_n\Phi_n(\zeta,\phi),$$

so that the boundary conditions are satisfied regardless of the Φ_n. The sequence in ζ is extended by simple polynomial expressions and that in ϕ by trigonometric expressions, both chosen to make the integration involved in the orthogonalization process as simple as possible without any loss in generality.

The eigenvalues, T_0 and corresponding wave numbers, k_0 arising from each representation (a) to (g) for u and v are summarized in the following table:

	a		*b*		*c*		*d*	
ϵ	k_0	T_0	k_0	T_0	k_0	T_0	k_0	T_0
0·0	3·117	3,499·9	3·128	3,474·9	3·117	3,499·9	3·128	3,474·9
0·1	3·103	3,561·7	3·116	3,524·2	2·912	3,190·6	2·922	3,150·2
0·2	3·060	3,763·9	3·086	3,681·2	2·728	3,068·7	2·746	2,986·9
0·3	2·995	4,168·4	3·041	3,977·2	2·541	3,143·7	2·596	2,990·5
0·4	2·912	4,932·3	2·995	4,478·8	2·409	3,503·1	2·473	3,200·7
0·5	2·820	6,505·8	2·971	5,332·1	2·259	4,441·5	2·388	3,756·7
0·6	2·722	10,784	3·014	6,880·8	2·120	7,410·3	2·395	5,108·3
0·7	2·623	49,313	3·246	10,030	$\sim$5	$\sim$38,000	2·836	8,921·9
0·8	—	—	$\sim$3·73	$\sim$17,200	—	—	3·957	17,171
0·9	—	—	—	—	—	—	$\sim$4·9	$\sim$35,000

	e		*f*		*g*	
ϵ	k_0	T_0	k_0	T_0	k_0	T_0
0·0	3·128	3,474·9	3·117	3,499·9	3·127	3,474·9
0·1	2·931	3,110·8	3·110	3,566·3	3·122	3,529·6
0·2	2·763	2,906·9	3·088	3,780	3·108	3,702·3
0·3	2·626	2,843·6	3·052	4,197·7	3·091	4,022·2
0·4	2·515	2,932·0	3·007	4,952·8	3·076	4,554·0
0·5	2·446	3,228·3	2·951	6,407·6	3·077	5,438·7
0·6	2·446	3,892·6	2·890	9,864·1	3·127	7,013·4
0·7	2·695	5,379·7	2·823	25,883	3·323	10,228
0·8	3·091	9,017·2	—	—	$\sim$3·8	$\sim$18,000
0·9	$\sim$4·0	$\sim$19,900	—	—	$\sim$5	$\sim$50,000

8. Discussion

As was mentioned previously, due to the symmetry in ϕ of the equations of the simplified eigenvalue problem, i.e. when the function F is neglected in equation (32*b*), symmetric and antisymmetric eigenfunctions exist completely independently, with corresponding independent eigenvalues. The marginal stability problem, however, is concerned with finding the minimum of the eigenvalue T at which an infinitesimal disturbance can exist. By comparing the eigenvalues from the symmetric forms (*a*) and (*b*) with those from the antisymmetric forms (*f*) and (*g*) it will be noted that the former are invariably the smaller for each eccentricity ratio, except at higher values of ϵ, where considering the simple trial representations used, the results are likely to be inaccurate. It can be implied therefore, that at marginal stability only symmetric disturbances will exist. This deduction is confirmed to some extent by the work of DiPrima (1963) who showed that the local clearance and pressure gradients have symmetric effects. It should be noted that if the function F, the terms of which may be shown to be antisymmetric, were to be included in the equation, then both types of disturbance would probably occur at marginal stability, although it appears that the antisymmetric part might be relatively unimportant. Clearly solutions of the problem when F is retained would be of great interest, but the problem then is extremely complex, and in fact, due to the way in which R and ψ appear, no Taylor number can be defined unless ψ is left as a variable parameter.

In figure 2 are plotted the critical Taylor numbers for each of the five symmetric cases (a) to (e), as functions of the eccentricity ratio. Each curve represents an estimate of the critical Taylor number at which a disturbance of a prescribed

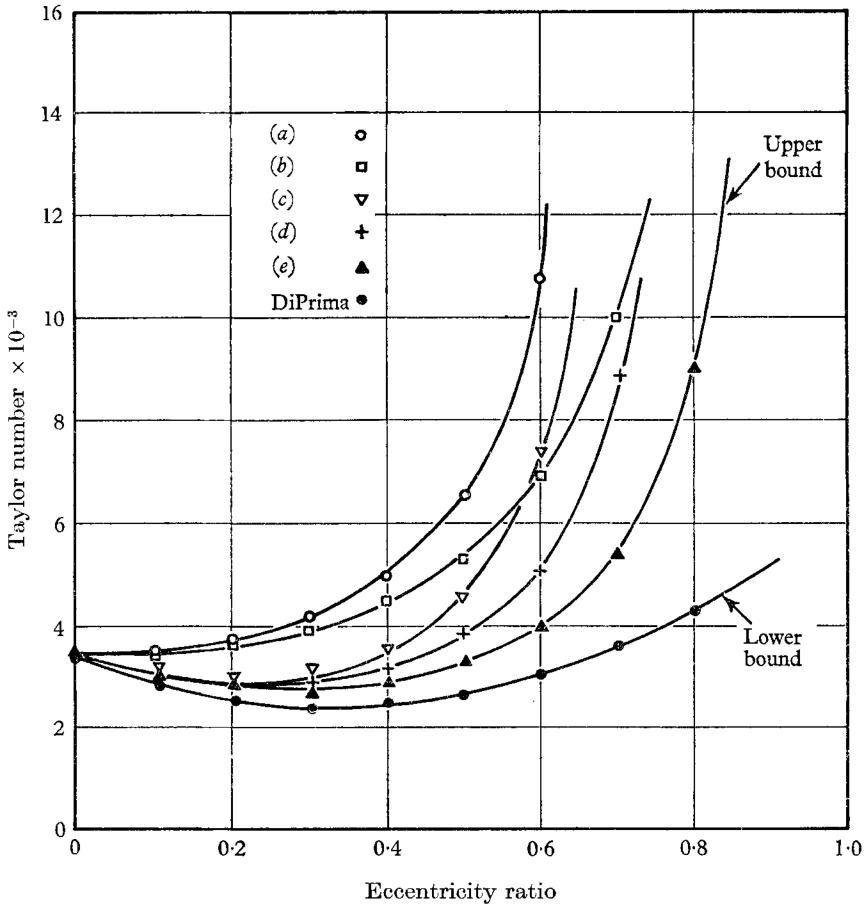

FIGURE 2. Variation of critical Taylor number with eccentricity ratio.

form can exist in a state of marginal stability. As more terms are added to the series representations for u and v, the resulting curves will tend to some limiting curve which is the true stability boundary of the problem under consideration. The curve (e) therefore, resulting from the solution of numerous 8×8 determinants, is the best estimate of the stability boundary which has been obtained using the method of this paper. Better estimates could, of course, be obtained by extending the series, but this would result in an increased order (and number) of determinants to be solved and a consequent prohibitive increase in the volume of calculation, particularly when it is recalled that the problem being considered is already approximate. However, it is considered that the curve (e) may always be regarded as an upper bound to the stability boundary of the simplified eigenvalue problem considered, i.e. that where F is neglected. Furthermore, it seems that it will also be an upper bound for the 'exact' small gap problem (29), (30) since, if the disturbance is constrained to be symmetric in ϕ, the antisymmetric terms comprising F do not influence the eigenvalue when Galerkin's method is used.

A sixth curve is plotted in figure 2, resulting from the work of DiPrima (1963). These results were obtained by estimating the critical Taylor number at the least stable position in a local sense, invariably $\phi = 0$, basing the calculation on the equivalent flow between concentric cylinders, using the local values of clearance and circumferential pressure gradient appropriate to the flow between eccentric cylinders. DiPrima concludes that this curve must represent the limit below which the flow is certain to be stable, and may therefore be considered as a strict lower bound for the stability boundary of the problem of this paper.

With the establishment of these upper and lower bounds, the position of the stability boundary seems fairly well defined for eccentricity ratios up to 0·6. In most of this range it will be noted that the critical Taylor number is smaller than that for $\epsilon = 0$, implying that for a considerable range of eccentricity ratio the flow is less stable than that between concentric cylinders. Until recently, experimental evidence regarding a destabilizing effect of eccentricity was somewhat inconclusive. The results of Cole, who used a suspension of fine aluminium particles as a flow visualization technique, indicate a decrease of stability at small eccentricities but a progressive increase beyond $\epsilon \sim 0{\cdot}2$. Kamal's results, using a similar technique show no destability and a progressive increase in stability with ϵ. Some recent experimental work by Castle & Mobbs (1967) however, throws some light on the discrepancy between the theory of this paper and the earlier experimental work. Using an apparatus in which dye was injected into the fluid through small orifices in the rotating inner cylinder, it was shown that a destabilizing effect of eccentricity did occur, although the vortex flow apparently remained of very small circulation for Taylor numbers considerably in excess of the critical. The stability boundary observed in this manner is in reasonable agreement with the theoretical upper bound (e) of figure 2. Furthermore, in the same apparatus, when the aluminium suspension technique was used instead of the dye injection technique no instability was detectable until Taylor numbers nearer those determined by Cole and Kamal, indicating that the former technique is not sufficiently sensitive to observe the primary instability.

In conclusion, since the present work is apparently the first in which the three-dimensional nature of the vortex flow between eccentric cylinders has been considered in a general manner, it appears worthwhile to discuss the approximations which have appeared necessary in tackling the problem. In the order in which they were introduced, the approximations are: (i) the small gap approximation; (ii) the neglect of the inertia terms in the laminar solution; (iii) the neglect of the antisymmetric terms in the eigenvalue problem; (iv) the use of Galerkin's method in an eigenvalue problem with two dependent variables.

The neglect of terms which are expected to be small in view of the small relative clearance is a common first approximation in stability analyses of this nature. The error involved, for the concentric case, can be directly determined from the literature. For $\psi = 0{\cdot}05$, the error in the eigenvalue T is $\sim 6\,\%$, while for $\psi = 0{\cdot}25$

the error is $\sim 31\,\%$, indicating that the use of the small gap approximation yields results for the eigenvalue in error by $O(\psi T)$.

The neglect of the inertia terms in the laminar solution, at first sight serious in view of the magnitude of $R\psi$, has been shown to yield errors probably smaller than those produced by the small gap approximation.

The neglect of the antisymmetric terms in the eigenvalue problem is undoubtedly the most serious approximation made, but one which it seemed necessary to make in order to obtain a solution of the problem. It has been justified to some extent for small eccentricity ratios and it is expected that the eigenvalues predicted in this range will be reasonably accurate. At larger eccentricities the accuracy must remain in doubt until the solution of the full equations is accomplished.

Finally, the use of the Galerkin method for the present eigenvalue problem is somewhat unorthodox. The author has been unable to prove analytically that its use in this case leads to a sequence of upper bounds for the eigenvalue as the approximate eigenfunctions are extended. However, the basis of the method, i.e. the relaxation of constraint on the eigenfunctions by increasing the number of free parameters, and the results obtained, both here and by DiPrima (1961), seem to indicate that such a minimizing sequence does occur.

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